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On the Disproof of Spectral Synthesis*

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1. INTRODUCTION

In 1959 Malliavin proved the impossibility of spectral synthesis for an arbitrary noncompact Abelian group G. The problem is easily reduced to the case where G is discrete (cf. [8], p. 173), and Malliavin succeeded in proving the impossibility for discrete G. His proof naturally breaks into two parts; concerning these Rudin remarks in [8].

"The first part contains the main idea.... The second part consists of a construction which, though not simple, is merely a matter of technique, and several possibilities exist. Following Kahane" (see[1, 4]) "we use a method based on probability considerations; this simplifies the required computations and also shows that, in a certain sense, 'randomly selected' compact sets fail to be S-sets."

However the resulting computations are still not very simple. They involve the estimation of a triple integral over $\Gamma \times \Gamma \times \Gamma$ (Γ being the dual of G) where the coefficients of the functions being integrated lie in a probability space.

The main purpose of this paper is to give an easier construction. This is described in Section 3: it is purely combinatorial, and requires no calculations of integrals or special functions; at the same time it eliminates the need for probability considerations. The idea is to look directly at

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the sequence of Fourier coefficients; rather than being very "messy", this approach appears to be the simplest one.

(Another simplified construction, based on a quite different idea, has been given by Kahane and Katznelson [3, 5].)

In Section 2 we consider the first part of Malliavin's proof; the discussion follows Rudin [8], with certain modifications. Our purpose here is to show that, after adopting a suitable point of view, the crucial identities become "obvious".

Finally, it should be noted that Varopoulos [9], [10] (cf. also [2]) has recently given a completely different proof of the impossibility of spectral synthesis, which is in many respects more elegant than Malliavin's. This proof uses the simple example of L. Schwartz for $G = R^3$, together with a series of constructions involving tensor products of Banach algebras.

However Malliavin's counterexample is interesting because it proves much more than was originally required. For instance, it follows immediately from his result that spectral synthesis fails even for principal ideals.

Notation and Definition of Terms

G is an infinite discrete Abelian group. Γ is its dual group which is compact and not discrete.

For $f \in L^1(G)$, the Fourier transform $\hat{f}(\gamma)$, $\gamma \in \Gamma$, is defined by $\hat{f}(\gamma) = \sum_{p \in G} f(p)\,(-p, \gamma)$.

For $f \in L^1(\Gamma)$ we set $\hat{f}(p) = \int_\Gamma f(\gamma)\,(p, \gamma) d\gamma$, so that the Fourier inversion theorem holds: $f^{\wedge\wedge} = f$.

The algebra of Fourier transforms $\hat{f}(\gamma), f \in L^1(G)$, is denoted by $A(\Gamma)$. We define $\| \hat{f} \|$ to be the L^1 norm of f, $\| \hat{f} \| \equiv \sum | f(p) |$, so that $A(\Gamma)$ is just an isomorphic and isometric copy of $L^1(G)$.

For $\hat{f} \in A(\Gamma)$, $\eta(\hat{f}) \equiv \sup_p | f(p) |$.

The *zero set* or *hull* of an ideal $I \subseteq A(\Gamma)$ is the set of points $\gamma_0 \in \Gamma$ such that $f(\gamma_0) = 0$ for all $f \in I$. The *spectral synthesis hypothesis* for G is the assertion that two ideals in $A(\Gamma)$ have the same zero set if and only if their closures in $A(\Gamma)$ coincide. Here the sufficiency is trivial, and the necessity is known for special cases, e.g., if the zero sets are empty (the Wiener tauberian theorem).

2. THE KEY THEOREM; MALLIAVIN'S OOPERATIONAL CALCULUS

First it is convenient to define, for $g \in A(\Gamma)$,

$$\eta(g) \equiv \sup_{p \in G} | \hat{g}(p) |. \tag{2.1}$$

Thus $\eta(g)$ is simply the L^∞ norm of the sequence $\hat{g}(p)$, whereas $\| g \|$ denotes the L^1 norm. It follows immediately that, given any $g, h \in A(\Gamma)$,

$$\left| \int_\Gamma g(\gamma)h(\gamma)d\gamma \right| \leqq \eta(g) \| h \|. \tag{2.2}$$

For, by Parseval's identity, $\int_\Gamma g(\gamma)h(\gamma)d\gamma = \Sigma_G \hat{g}(p)\hat{h}(-p)$.

The following is an extension due to Rudin [7] of a theorem of Malliavin [6]. Since every power of $(f - \alpha)$ has the same zero set in Γ, this result gives a counterexample to spectral synthesis in $L^1(G)$ provided there exists a suitable function f.

THEOREM 1. *Let f be a real-valued function in $A(\Gamma)$. Suppose that, with* $-\infty < u < \infty$,

$$\eta[\exp(iuf)] = O[| u |^{-n}] \text{ for all } n \geqq 0. \tag{A}$$

Then, for some real constant α, the closed ideals in $A(\Gamma)$ generated by $(f - \alpha)^n$, $n = 1, 2, \ldots$, are all distinct.

REMARKS. Denote by $a_p(u)$ the p-th Fourier coefficient of $\exp[iuf(\gamma)]$ so that $\exp[iuf(\gamma)] = \Sigma_{p \in G}\, a_p(u)\,(-p, \gamma)$. Then (A) asserts that the function $a^*(u) \equiv \sup_{p \in G} | a_p(u) |$ is $O(| u |^{-n})$ for all $n \geqq 0$. Thus for each n, $\int_{-\infty}^{\infty} \sup | a_p(u) | \; | u |^n du$ exists and is finite [the function $a^*(u)$ is continuous because $\eta(\;\;) \leqq \|\;\;\|$ and $\exp(iuf)$ is continuous in $A(\Gamma)$].

Proof. Let K be the range of f; K is a compact set in the real line. Let X denote the Banach space of continuous complex functions on K with the usual sup norm.

For each element $g \in A(\Gamma)$, define a bounded linear functional S_g on X by

$$S_g(\varphi) \equiv \int_\Gamma \varphi(f(\gamma))g(\gamma)d\gamma, \quad \varphi \in X. \tag{2.3}$$

Then, by the Riesz representation theorem, there exists for each g a unique bounded complex measure μ_g on K such that

$$\int_\Gamma \varphi(f(\gamma))g(\gamma)d\gamma = \int_{-\infty}^{\infty} \varphi(t)d\mu_g(t). \tag{2.4}$$

Morevoer, μ_g is linear in its dependence on g. (However this does not hold for f, which we regard as being held constant.)

From the definition of μ_g we immediately derive a basic identity: let $\varphi(t)$ be analytic, let $g \in A(\Gamma)$, and let $h(\gamma) \equiv \varphi(f(\gamma))g(\gamma)$ [so that $h \in A(\Gamma)$]. Then

$$\mu_h(t) = \varphi(t)\mu_g(t). \tag{2.5}$$

(The proof is clear.) Now for the first time we consider a result which depends on (A).

LEMMA. *For all $g \in A(\Gamma)$, the measure μ_g is of the form $m_g(t)dt$, where $m_g \in C^\infty$; moreover,*

$$| D^n m_g(t) | \leqq M_n \| g \| \qquad (all\ t). \tag{2.6}$$

where M_n is a constant independent of g. (Here D^n denotes the n-th derivative.)

Proof. It follows from (2.4) that the inverse Fourier transform of μ_g, $\hat{\mu}_g(u) \equiv \int_{-\infty}^{\infty} \exp(iut)d\mu_g(t)$, is equal to $\int_\Gamma \exp[iuf(\gamma)]g(\gamma)d\gamma$. Hence by (2.2), $| \hat{\mu}_g(u) | \leqq \eta[\exp(iuf)] \| g \|$, $-\infty < u < \infty$.

Set $M_n = \int_{-\infty}^{\infty} \eta[\exp(iuf)] \, | u |^n du$; Condition (A) gives $M_n < \infty$ for all n. Comparing this with the above we have $\int_{-\infty}^{\infty} | u^n \hat{\mu}_g(u) | \, du \leqq M_n \| g \|$. The desired result follows from the standard Fourier inversion theorem applied to μ_g and $\hat{\mu}_g$. Q.E.D.

Now let $\mu_0(t) = m_0(t)dt$ be the measure μ_g corresponding to $g(\gamma) \equiv 1$. Since $\int_{-\infty}^{\infty} m_0(t)dt = \int_\Gamma 1 \, d\gamma = 1$, $m_0(t)$ cannot be identically zero. Let α be any real number for which $m_0(\alpha) \neq 0$. We will show that the closed ideals in $A(\Gamma)$ generated by $(f - \alpha)^n$, $n = 1, 2, \ldots$, are distinct.

Define the linear functional T_n on $A(\Gamma)$ by

$$T_n(g) \equiv D^n[m_g(t)](\alpha). \tag{2.7}$$

It follows from (2.6) that T_n is bounded in terms of the norm on $A(\Gamma)$.

From (2.5) and (2.7) we obtain, for any polynomial $\varphi(t)$ and any $g \in A(\Gamma)$

$$T_n[\varphi(f(\gamma))g(\gamma)] = D^n[\varphi(t)m_g(t)](\alpha). \tag{2.8}$$

Setting $\varphi(t) = (t - \alpha)^{n+1}$ gives $T_n[(f - \alpha)^{n+1}g] = 0$ for all g; setting $\varphi(t) = (t - \alpha)^n$, $g \equiv 1$, gives $T_n[(f - \alpha)^n] = n!m_0(\alpha) \neq 0$.

These calculations, together with the fact that T_n is bounded, show that T_n annihilates the closed ideal generated by $(f - \alpha)^{n+1}$. But $T_n[(f - \alpha)^n] \neq 0$, whence $(f - \alpha)^n$ does not belong to this ideal.

The proof is complete.

3. The Construction of f.

Our next result holds for an arbitrary infinite discrete group G, but the proof will be given only for the case where G is the group of integers (Γ is the circle group). This makes certain of the lemmas easier to state and prove, without sacrificing any of the main ideas. The general case will be discussed in the next section.

THEOREM 2. *There exists a function which satisfies the hypotheses of the preceding theorem. In fact, somewhat more can be proved: namely, given $s > 1$, there is a real-valued function $f \in A(\Gamma)$ such that*

$$\eta[\exp(iuf)] = O[\exp(-\delta \mid u \mid^{1/s})] \text{ for some } \delta > 0. \tag{B}$$

[*As above, $\eta(g)$ denotes $\sup_p \mid \hat{g}(p) \mid$.*] *Here f has the form*

$$f(\gamma) = \sum_{k=1}^{\infty} \frac{1}{k^s} \operatorname{Re}(p_k, \gamma), \tag{3.1}$$

where $\{p_k\}$ is a suitably chosen sequence of elements from the group G.

Remark. It can be shown, however, that there is no real $f \in A(\Gamma)$ for which $\eta[\exp(iuf)] = O[\exp(- \delta \mid u \mid)]$ $(\delta > 0)$.

To see this, let $\eta_r(u)$, $1 \leq r \leq \infty$, be the L^r norm of the sequence $[\exp(iuf(\gamma))]^\wedge$. [I.e., $\eta_r(u) = \{\sum_{p \in G} \mid a_p(u) \mid^r\}^{1/r}$ where $a_p(u) = \int_\Gamma \exp(iuf(\gamma))\,(p, \gamma)d\gamma$.] Then, since $\mid \exp(iuf) \mid \equiv 1$, $\eta_2(u) \equiv 1$. Moreover, by the spectral radius formula, $\lim_{u \to +\infty} [\eta_1(u)]^{1/u} = 1$, so that $\eta_1(u)/\exp(\delta u) \to 0$ $(u \to + \infty)$ for all $\delta > 0$. But a very weak form of the Riesz convexity theorem gives $\eta_2(u)^2 \leqq \eta_1(u)\eta_\infty(u)$, whence $\eta_\infty(u)/\exp(- \delta u) \to \infty$ for all $\delta > 0$.

Proof of theorem. For the sake of clarity, *we will assume that G is the additive group of integers.*

Let us also take $s = 2$ in (3.1), so that what we wish to show is

$$\eta[\exp(iuf)] = O(\exp(-\delta\sqrt{|u|})] \quad (\delta > 0). \tag{B$'$}$$

We begin by considering some special properties of the norm η. The results which we need are contained in (a), (b), (c) to follow, and in Lemma 1.

In the first place, $\eta(g + h) \leqq \eta(g) + \eta(h)$ and $\eta(ag) = |a| \eta(g)$ for constant a. However η is not multiplicative—in general $\eta(gh)$ may be greater than $\eta(g)\eta(h)$. Clearly,

$$\eta(g) \leqq \| g \|; \tag{3.2}$$

that is, $\sup | \hat{g}(p) | \leqq \Sigma | \hat{g}(p) |$.

(a) If $h \in A(\Gamma)$, $| h | \leqq 1$, then $\eta(h) < 1$ unless h is a constant times a character [i.e., unless $h(\gamma) = \alpha \cdot (p_0, \gamma)$ where $p_0 \in G$ and α is a constant, $| \alpha | = 1$].

For the proof, observe that since $| h | \leqq 1$, the L^2 norm of $h(\gamma)$ is $\leqq 1$, whence $\Sigma | \hat{h}(p) |^2 \leqq 1$.

(b) The norm η is continuous in terms of the topology on $A(\Gamma)$. This is simply a consequence of (3.2).

(c) For all integers $k \neq 0$, the norm $\eta[g(k\gamma)]$ is independent of k ($g \in A(\Gamma)$). (Of course $k\gamma$ means $\gamma + \cdots + \gamma$ (k times) if $k > 0$; similarly for $k < 0$.)

This holds because G is the group of integers. Thus the Fourier coefficients of $g(k\gamma)$ are exactly the same as those of $g(\gamma)$, except that the sequence $[g(k\gamma)]^\wedge$ contains gaps: if $g(\gamma) = \Sigma\, g_p \exp(-ip\gamma)$ then $g(k\gamma) = \Sigma\, g_p \exp(-ipk\gamma)$.

The next lemma asserts that η is "nearly multiplicative" for products $g(\gamma) \cdot h(k\gamma)$ provided the integer k is sufficiently large. The hypotheses on g_a, h_a are motivated by the fact that analytic functions such as $f_a(\gamma) = \exp(ia \cos \gamma)$, $a \in (-\infty, \infty)$, define continuous mappings from $(-\infty, \infty)$ into $A(\Gamma)$. It is the application of this lemma which determines the choice of the sequence $\{p_k\}$.

LEMMA 1. *Let g_a and h_a be nonzero elements of $A(\Gamma)$ which depend continuously on a real parameter a. Then for any $M > 0$ and any $\varepsilon > 0$, there exists a positive integer k such that*

$$\eta[g_a(\gamma) \cdot h_a(k\gamma)] < (1 + \varepsilon) \cdot \eta(g_a) \cdot \eta(h_a) \text{ for all } a \in [-M, M]. \quad (3.3)$$

[Because of (c), $\eta(h_a)$ may be interpreted either as $\eta(h_a(\gamma))$ or $\eta(h_a(k\gamma))$.]

Proof. Since $\eta(g_a)$ and $\eta(h_a)$ are bounded away from zero when $a \in [-M, M]$, Eq. (3.3) may be deduced from the simpler inequality $\eta(g_a(\gamma) \cdot h_a(k\gamma)) \leqq \eta(g_a) \cdot \eta(h_a) + \varepsilon$. This is what we shall prove.

Let K be an upper bound for the values of $\eta(h_a)$,$a \in [-M, M]$. Now $\sum \hat{g}_a(p)$ is absolutely convergent, and we can find an N such that, for all $a \in [-M, M]$

$$\sum_{|p|>N} | \hat{g}_a(p) | < \varepsilon/K. \quad (3.4)$$

To see this, set $w_n(a) \equiv \sum_{|p|>n} | \hat{g}_a(p) |$. Then $w_n(a)$ is continuous in a, $w_n(a) \geqq w_{n+1}(a)$, and $w_n(a) \to 0$ as $n \to \infty$. Thus by Dini's theorem, $w_n(a) \to 0$ uniformly for $a \in [-M, M]$.

Take any $k > 2N$. Let $\{a_p\}$, $\{b_p\}$, $\{c_p\}$ be the Fourier coefficients of $g(\gamma)$, $h(k\gamma)$, and $g(\gamma)h(k\gamma)$, respectively. Recall that (i) $c_p = \sum_q a_q b_{p-q}$, and (ii) $b_p = 0$ except when k divides p.

Now to estimate $| c_p |$. Since $k > 2N$, only one nonzero term $a_q b_{p-q}$ in the sum (i) will come from the interval $-N \leqq q \leqq N$. The absolute value of this term is $\leqq \eta(g_a)\eta(h_a)$. By (3.4) the sum of the other terms is in absolute value less than (ε/K) $(\sup | b_p |) = (\varepsilon/K)\eta(h_a) \leqq \varepsilon$. Therefore $| c_p | \leqq \eta(g_a)\eta(h_a) + \varepsilon$ for all p. Q.E.D.

We are now in a position to attack our original problem. Let $f(\gamma) \equiv \sum_{k=1}^{\infty} (1/k^2)\mathrm{Re}(p_k, \gamma)$, where the sequence $\{p_k\}$ is yet to be determined. Then

$$\exp(iuf(\gamma)) = \prod_{k=1}^{\infty} b_k(\gamma, u), \quad (3.5)$$

where

$$b_k(\gamma, u) \equiv \exp\left[\frac{iu}{k^2} \cos p_k\gamma\right]. \quad (3.6)$$

Thus we need to estimate the η norm the of infinite product (3.5). Lemma 1 will allow us to compare this to the product of the norms of expressions like those in (3.6). First we need another lemma.

LEMMA 2. *For any closed interval* $[a, b]$ *not containing* 0, *there exists a constant* $\varrho < 1$ *such that* $\eta(\exp(ia \cos k\gamma)) \leqq \varrho$ *for all* $k \neq 0$ *and all* $a \in [a, b]$.

Proof. By (c) above, this norm is independent of k ($k \neq 0$). Now $\exp(i\alpha \cos k\gamma)$ is not equal to a constant times a character unless $\alpha = 0$. Hence the desired result follows from (a) and (b) (a continuous function on a compact set assumes a maximum value). Q.E.D.

Define $c_n(\gamma)$ to be $\prod_{k=1}^{n} b_k(\gamma)$, where, as above, $b_k(\gamma, u) = \exp[(iu/k^2) \cos p_k\gamma]$. Let $\{\varepsilon_k\}$, $\varepsilon_k > 0$, be chosen so that $\prod_{k=1}^{\infty} (1 + \varepsilon_k) < 2$.

Now using Lemma 1, we can define $\{p_n\}$ inductively so that for each n

$$\eta(c_{n+1}) < (1 + \varepsilon_n)\eta(c_n)\eta(b_{n+1}) \quad \text{for} \quad |u| \leqq n^2. \tag{3.7}$$

That is, $c_n(\gamma, u)$ being given, the positive integer p_{n+1} and hence the function $b_{n+1}(\gamma, u)$ are determined so that (3.7) holds. Note that the range of admissible values of u increases with each n.

From (3.7) we obtain

$$\eta\left(\prod_{k=1}^{\infty} b_k\right) < 2\,\eta(c_n) \prod_{k>n} \eta(b_k) \quad \text{whenever} \quad |u| \leqq n^2. \tag{3.8}$$

[Passing to the limit in the infinite product is justified because η is continuous on $A(\Gamma)$.]

Now partition the interval $[1, \infty)$ into sections $4^{n-1} \leqq |u| \leqq 4^n$ and break the sequence of integers into segments $2^n < k \leqq 2^{n+1}$.

By lemma 2, with $[a, b] = \pm [\frac{1}{16}, 1]$, there exists a constant $\varrho < 1$ such that $4^{n-1} \leqq |u| \leqq 4^n$, $2^n < k \leqq 2^{n+1}$ implies

$$\eta(b_k) = \eta(\exp[(iu/k^2) \cos p_k\gamma]) \leqq \varrho. \tag{3.9}$$

Since we always have $\eta(b_n), \eta(c_n) \leqq 1$ [because $|b_n(\gamma)| = |c_n(\gamma)| = 1$], it follows from (3.8) and (3.9) that

$$\eta\left(\prod_{k=1}^{\infty} b_k\right) < 2\varrho^{2^n} \quad \text{for} \quad 4^{n-1} \leqq |u| \leqq 4^n. \tag{3.10}$$

(Proof: replace n by 2^n in (3.8) and consider only the terms $\eta(b_k)$ with $2^n < k \leqq 2^{n+1}$.)

Finally, (3.10) is equivalent to (B′): $\eta[\prod_{k=1}^{\infty} b_k] = \eta[\exp(iuf)] = O[\varrho^{|u|^{1/2}}]$ with $\varrho < 1$. This proves Theorem 2.

4. The Case of an Arbitrary Discrete Group G

In the above proof we assumed that G was the group of integers. However only a few modifications are needed for the general case.

Specifically, the assertion (c) is false, and Lemmas 1 and 2 require modification, both in their statement and proof. On the other hand, the results preceding (c) still hold, and the use of Lemmas 1 and 2 in the rest of the argument remains essentially unchanged.

We state the proper generalizations of Lemmas 1 and 2, and give a brief sketch of their proofs. The extra steps needed are exactly like those in Rudin [8] (cf. the proof of (12) and (13) on pp. 179–80).

LEMMA 1a. *Let $g_\alpha \in A(\Gamma)$ depend continuously on a real parameter α. Assume the same thing for $h_\alpha \in A(T)$, where T is the circle group. Then for any $M > 0$ and any positive ε, there exists an element $p \neq 0$ in G such that for all $\alpha \in [-M, M]$*

$$\eta(g_\alpha(\gamma) \cdot h_\alpha[(p, \gamma)]) \leqq \eta(g_\alpha(\gamma)) \cdot \eta(h_\alpha[(p, \gamma)]) + \varepsilon. \tag{4.1}$$

LEMMA 2a. *For any closed interval $[a, b] \subset (0, \pi/2)$, there exists a constant $\varrho < 1$ such that $\eta(\exp[i\alpha \operatorname{Re}(p, \gamma)]) \leqq \varrho$ for all $p \neq 0$ in G and all $\alpha \in [a, b]$.*

Sketch of the proofs. For Lemma 2a: In the first place, the value of $\eta(\quad)$ depends only on α and on the order r of the element p in G. Write this as $\eta(\alpha, r)$ ($r = 2, 3, \ldots, \infty$). For fixed α, as $r \to \infty$, $\eta(\alpha, r) \to \eta(\alpha, \infty)$.

(*Proof.* Let $\sum_{p=-\infty}^{\infty} a_p \exp(-ip\theta)$ be the ordinary Fourier series of the function $\exp(i\alpha \cos\theta)$, $\theta \in [0, 2\pi]$. Then $\eta(\alpha, \infty) = \sup | a_p |$, whereas $\eta(\alpha, r) = \max \{ | \sum_{k=-\infty}^{\infty} a_{rk+j} | \mid 0 \leqq j < r\}$. But $\sum_{p=-\infty}^{\infty} a_p(\alpha)$ is absolutely convergent, uniformly in α [cf. (3.4)].)

The conditions $p \neq 0$, $0 < \alpha < \frac{1}{2}\pi$ insure that $h(\gamma) \equiv \exp[i\alpha \operatorname{Re}(p, \gamma)]$ cannot be a constant times a character (because h is not constant and its values lie inside a semicircle). Thus, by (a) in the preceding section, $\eta(\alpha, r) < 1$ for fixed α, r. The desired uniformity can be proved by standard arguments [$\eta(\alpha, r)$ is a continuous function of α, and as $r \to \infty$, $\eta(\alpha, r) \to \eta(\alpha, \infty)$ uniformly in α].

For the proof of lemma 1a: Following Malliavin, we consider three cases.

Case 1. There exists a finite upper bound for the orders of the elements in G.

Case 2. Every element in G has finite order, but these orders are not bounded.

Case 3. G contains an element of infinite order.

Now case 3 can be reduced to the situation discussed in the previous section, since G contains a subgroup isomorphic to the integers.

Case 2 is similar to 3: for all but a finite set $\{p_k\} \subseteq G$, the corresponding Fourier coefficients of $g_a(\gamma)$ will be "negligibly small". The element $p \in G$ is chosen to be of a "very much larger order" than any of the p_k. (cf. the proof of Lemma 1.)

However case 1 uses a completely different idea. Here it can be shown that G is an infinite direct sum of cyclic groups $G_1 \oplus G_2 \oplus \cdots$. Take an infinite sequence $\{p_n\}$, $p_n \neq 0$, with $p_1 \in G_1$, $p_2 \in G_2$, etc. Now if the function g_a depends only on $p_1, \ldots, p_n$, and h_a on $p_{n+1}, p_{n+2}, \ldots$, then $\eta(g_a h_a) \leqq \eta(g_a)\eta(h_a)$.

This gives a sharper inequality than Lemma 1a (no "ε") but for a more restricted class of functions g_a. Actually Lemma 1a is still true, but the simpler result above suffices for our purposes [cf. the inequalities (3.7) and (3.8)].

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